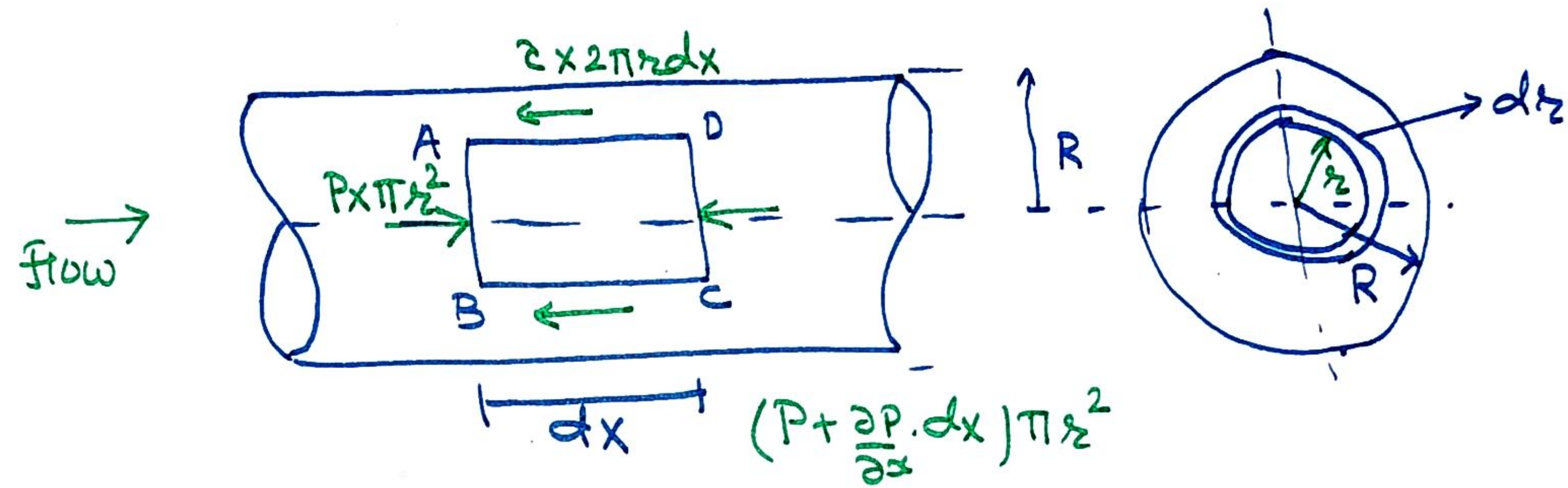
Flow of Viscous Fluid Through Circular Pipe- Hagen Poiseuille Law

Assumption

1) fluid follows Newton's law of viscosity

2) No slip of fluid at boundary



As There is no acceleration Resultant force is zero

$$P\pi r^2 - \left(P + \frac{\partial P}{\partial x}.dx\right)\pi r^2 - \tau\, 2\pi r dx = 0$$

$$P\cancel{\pi} r^2 - P\cancel{\pi} r^2 - \frac{\partial P}{\partial x}.dx\,\pi r^2 - 2\pi r dx\, \tau = 0$$

$$-\frac{\partial P}{\partial x}.\cancel{dx}\,\cancel{\pi} r^{\cancel{2}} = 2\cancel{\pi}\cancel{r}\cancel{dx}\,\tau \qquad -\frac{\partial P}{\partial x} = \frac{2\tau}{r}$$

$$-\frac{\partial p}{\partial x} = \frac{2\tau}{r}$$

$$\boxed{\tau = -\frac{\partial p}{\partial x}\cdot r/2}$$

$\frac{\partial p}{\partial x}$ is constant

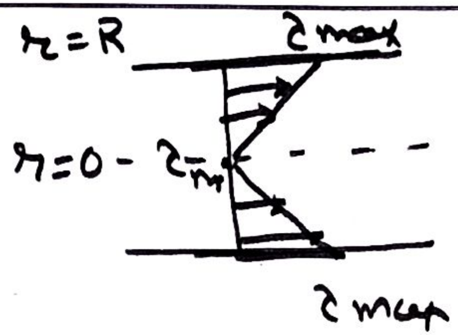
$\tau \rightarrow r$

$r = 0 \rightarrow$ para- located at centre of pipe

$$\boxed{\tau = -\frac{\partial p}{\partial x}\cdot 0 = 0}$$

$r = R$

$$\boxed{\tau = -\frac{\partial p}{\partial x}\cdot \frac{R}{2}}$$



(a) Velocity Distribution.

$$\tau = \mu \frac{\partial u}{\partial y}$$

y is measured from pipe wall

$$y = R - r$$

$$dy = 0 - dr$$

$$dy = -dr$$

$$\tau = \mu \frac{\partial u}{-\partial r}$$

$$\tau = -\frac{\partial p}{\partial x}\cdot r/2 \quad \text{— (a)}$$

$$\tau = -\mu \frac{\partial u}{\partial r} \quad \text{— (b)}$$

$$-\mu \frac{\partial u}{\partial r} = -\frac{\partial p}{\partial x}\cdot r/2$$

$$\frac{\partial u}{\partial r} = \frac{1}{2\mu}\frac{\partial p}{\partial x} r$$

integrating w.r.t. r

$$\int \frac{\partial u}{\partial r}\, dr = \int \frac{1}{2\mu}\frac{\partial p}{\partial x}\, dr$$

$$u = \frac{1}{2\mu}\cdot\frac{\partial p}{\partial x}\cdot\frac{r^2}{2} + C$$

boundary condition

$r = R \quad u = 0$

$$0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}\cdot\frac{R^2}{2} + C$$

$$C = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\cdot R^2/2$$

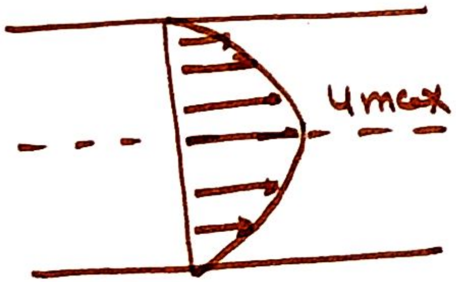
$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\cdot\frac{r^2}{2} - \frac{1}{2\mu}\frac{\partial p}{\partial x}\cdot R^2/2$$

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\cdot\left(\frac{r^2-R^2}{2}\right)$$

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}\cdot(r^2-R^2)$$

$\mu, \frac{\partial p}{\partial x}, R$ are Constant

$u \to r^2 \to$ Parabola



(b) Ratio of max. velocity to Average velocity

$r = 0$

$u = u_{max}$

$$u_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\cdot R^2$$

$\bar{u}$ = avg. velocity

dq = velocity at radius r × area

$= u \times 2\pi r\,dr$

$$\int_0^R dq = \int_0^R -\frac{1}{4\mu}\frac{\partial p}{\partial x}(R^2 - r^2)\,2\pi r\,dr$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}\,2\pi\int_0^R (R^2 - r^2)\cdot r\,dr$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}\cdot 2\pi\int_0^R (R^2 r - r^3)\,dr$$

$$Q = -\frac{1}{4\mu}\frac{\partial p}{\partial x} \cdot 2\pi \left[\frac{R^2 \cdot r^2}{2} - \frac{r^4}{4}\right]_0^R$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x} 2\pi \left[\frac{R^2 \cdot R^2}{2} - \frac{R^4}{4}\right]$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x} \cdot 2\pi \frac{R^4}{4}$$

$$Q = \frac{\pi}{8\mu} -\frac{\partial p}{\partial x} \cdot R^4$$

average velocity $\bar{u} = \frac{Q}{A}$

$$\bar{u} = \frac{\cancel{\pi}/8\mu \; -\frac{\partial p}{\partial x} \cdot R^{\cancel{4}}}{\cancel{\pi} R^{\cancel{2}}}$$

$$\bar{u} = \frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) \cdot R^2$$

$$\frac{u_{max}}{\bar{u}} = \frac{-\frac{1}{4\mu}\frac{\partial p}{\partial x} \cdot R^2}{\frac{1}{8\mu} -\frac{\partial p}{\partial x} \cdot R^2}$$

$$\frac{u_{max}}{\bar{u}} = 2$$

$\rightarrow$ $\bar{u} = u$

$$u = u_{max}\left[1 - (r/R)^2\right]$$

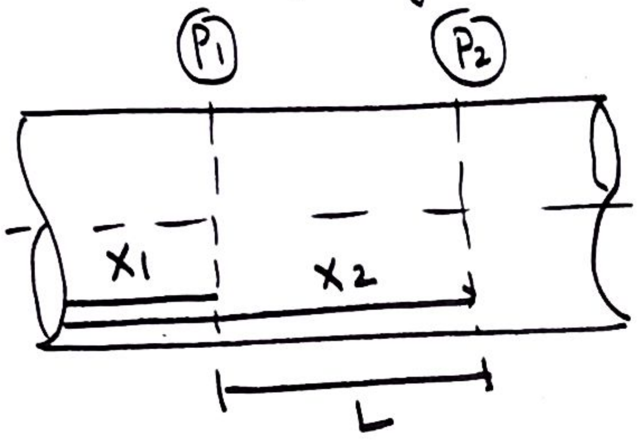
$$\bar{u} = \frac{u_{max}}{2}$$

$$u = \bar{u}$$

$$u_{max}\left[1 - (r/R)^2\right] = \frac{u_{max}}{2}$$

$$1 - (r/R)^2 = \frac{1}{2} \qquad \boxed{r = 0.707R}$$

(c) Drop of pressure for given length of pipe



$$\bar{u} = \frac{1}{8\mu} \cdot -\frac{\partial p}{\partial x} \cdot R^2$$

$$-\frac{\partial p}{\partial x} = \frac{8\mu\bar{u}}{R^2}$$

integrating w.r.t x

$$-\int_2^1 dp = \int_2^1 \frac{8\mu\bar{u}}{R^2}\,dx$$

$$-(P_1 - P_2) = \frac{8\mu\bar{u}}{R^2}(x_1 - x_2)$$

$$P_1 - P_2 = \frac{8\mu\bar{u}}{R^2}(x_2 - x_1)$$

$$x_2 - x_1 = L$$

$$P_1 - P_2 = \frac{8\mu\bar{u}}{R^2} \cdot L$$

$$R = D/2$$

$$P_1 - P_2 = \frac{8\mu\bar{u}L}{(D/2)^2}$$

$$P_1 - P_2 = \frac{32\mu\bar{u}L}{D^2}$$

$$h_f = \frac{P_1 - P_2}{\rho g}$$

$$\boxed{h_f = \frac{32\mu\bar{u}L}{D^2 \cdot \rho g}}$$