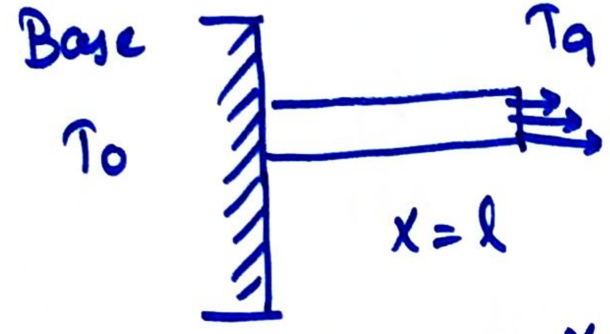
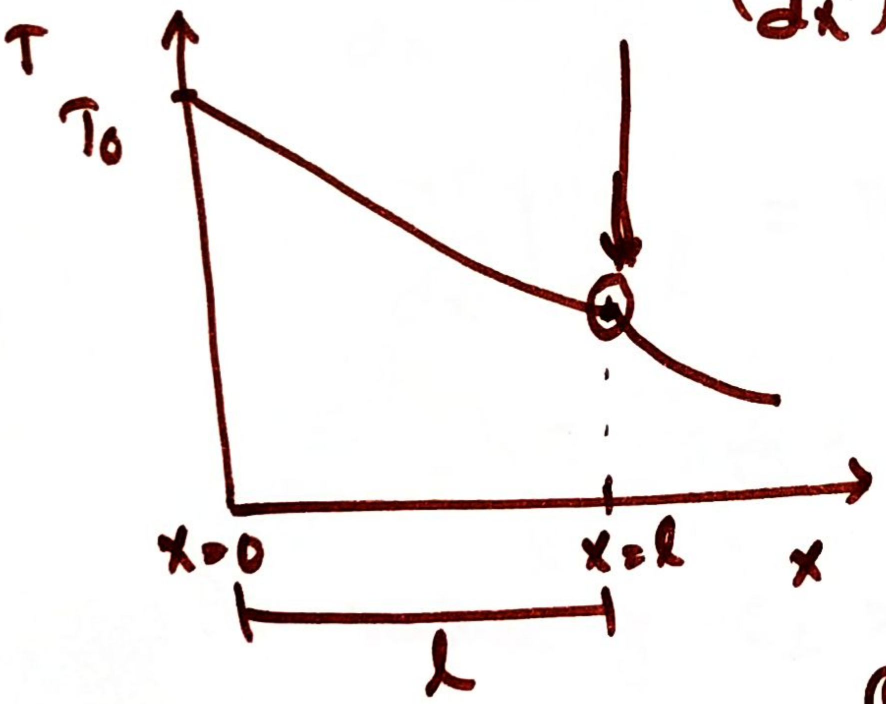
Case-III Heat Dissipation from a fin losing heat at tip



$x = l$ heat flow due to conduction

= heat flow by convection at tip

$$-K A_{cs}\left(\frac{dT}{dx}\right)_{x=l} = h A_{su}(T - T_a)$$



1) $x=0 \quad T=T_0$

$\theta = \theta_0$

2) $x = l$

$$-K A_{cs}\left(\frac{dT}{dx}\right)_{x=l} = h A_{su}(T - T_a)$$

$$\theta = c_1 e^{mx} + c_2 e^{-mx}$$

$$\theta = T - T_a = c_1 e^{mx} + c_2 e^{-mx}$$

$$\frac{d}{dx}(T - T_a) = \frac{d}{dx}(c_1 e^{mx} + c_2 e^{-mx})$$

$$\frac{dT}{dx} = c_1 m e^{mx} + c_2 - m e^{-mx}$$

$$\frac{dT}{dx} = m c_1 e^{mx} - m c_2 e^{-mx}$$

$$\left(\frac{dT}{dx}\right)_{x=l} = m c_1 e^{ml} - m c_2 e^{-ml}$$

$$-\frac{h\theta}{K} = m c_1 e^{ml} - m c_2 e^{-ml} \quad \text{— II}$$

By value of $c_2 = \theta_0 - c_1$ from eqn I

Applying boundary condition.

1) $X=0 \quad \theta=\theta_0 \quad T=T_0$

$$\theta = T - T_a = c_1 e^{mx} + c_2 e^{-mx}$$

$x=0$

$$\theta_0 = T_0 - T_a = c_1 e^{m(0)} + c_2 e^{-m(0)}$$

$$\boxed{c_1 + c_2 = \theta_0} \text{ --- } \mathrm{I} \qquad c_2 = \theta_0 - c_1$$

2) $$-K\cancel{A_{cs}}\left(\frac{dT}{dx}\right)_{x=l} = h\cancel{A_{su}}(T-T_a)$$

$A_{cs} = A_{su} \qquad T - T_a = \theta$

$$\left(\frac{dT}{dx}\right)_{x=l} = -\frac{h\theta}{K}$$

$$-\frac{h\theta}{k} = m c_1 e^{ml} - m c_2 e^{-ml}$$

$$c_2 = \theta_0 - c_1$$

$$\theta_l = c_1 e^{ml} + c_2 e^{-ml}$$

$$-\frac{h}{k}\left(c_1 e^{ml} + c_2 e^{-ml}\right) = m c_1 e^{ml} - m c_2 e^{-ml}$$

$$-\frac{h}{k}\left(c_1 e^{ml} + (\theta_0 - c_1) e^{-ml}\right) = m c_1 e^{ml} - m(\theta_0 - c_1) e^{-ml}$$

$$-\frac{h}{k} c_1 e^{ml} + \frac{-h}{k}\theta_0 e^{-ml} + \frac{h c_1}{k} e^{-ml} = m c_1 e^{ml} - m\theta_0 e^{-ml} + m c_1 e^{-ml}$$

$$-\frac{h}{k}c_1 e^{ml} + \frac{h}{k}c_1 e^{-ml} - mc_1 e^{ml} - mc_1 e^{-ml}$$

$$= \frac{h}{k}\theta_0 e^{-ml} - m\theta_0 e^{-ml}$$

$$-\frac{h}{mk}c_1 e^{ml} + \frac{h}{mk}c_1 e^{-ml} - c_1 e^{ml} - c_1 e^{-ml}$$

$$= \frac{h}{mk}\theta_0 e^{-ml} - \theta_0 e^{-ml}$$

$$c_1\left[(e^{ml} + e^{-ml}) + \frac{h}{mk}(e^{ml} - e^{-ml})\right] = \theta_0 e^{-ml}\left[1 - \frac{h}{mk}\right]$$

$$c_1 = \frac{\theta_0\left[1 - \frac{h}{mk}\right]e^{-ml}}{(e^{ml} + e^{-ml}) + \frac{h}{mk}(e^{ml} - e^{-ml})}$$

$$C_2 = \theta_0 - \frac{\theta_0\left[1-\frac{h}{mk}\right]e^{-ml}}{(e^{ml}+e^{-ml})+\frac{h}{mk}(e^{ml}-e^{-ml})}$$

$$C_2 = \frac{\theta_0\left(1+\frac{h}{mk}\right)e^{ml}}{(e^{ml}+e^{-ml})+\frac{h}{mk}(e^{ml}-e^{-ml})}$$

$$\theta = C_1 e^{mx} + C_2 e^{-mx}$$

$$\theta = \frac{\theta_0\left[1-\frac{h}{mk}\right]e^{-ml}}{(e^{ml}+e^{-ml})+\frac{h}{mk}(e^{ml}-e^{-ml})} \cdot e^{mx} + \frac{\theta_0\left[1+\frac{h}{mk}\right]e^{ml}}{(e^{ml}+e^{-ml})+\frac{h}{mk}(e^{ml}-e^{-ml})} \cdot e^{-mx}$$

$$\frac{\theta}{\theta_0} = \frac{\left[e^{m(l-x)} + e^{-m(l-x)}\right] + \frac{h}{mK}\left[e^{m(l-x)} - e^{-m(l-x)}\right]}{\left(e^{ml} + e^{-ml}\right) + \frac{h}{mK}\left(e^{ml} - e^{-ml}\right)}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\frac{\theta}{\theta_0} = \frac{\cosh[m(l-x)] + \frac{h}{mK}\sinh[m(l-x)]}{\cosh(ml) + \frac{h}{mK}\sinh(ml)}$$

$$q_{fin} = -K A_{cs} \left(\frac{dT}{dx}\right)_{x=0}$$

$$T - T_a = (T_o - T_a) \left[\frac{\cosh m(l-x) + \frac{h}{mK} \sinh m(l-x)}{\cosh ml + \frac{h}{mK} \sinh ml} \right]$$

$$\frac{dT}{dx} = (T_o - T_a) \frac{-m \sinh m(l-x) - m\left[h/mK \cosh m(l-x)\right]}{\cosh ml + \frac{h}{mK} \sinh ml}$$

$$\left(\frac{dT}{dx}\right)_{x=0} = (T_o - T_a)\, m \left[\frac{\sinh ml + \frac{h}{mK} \cosh ml}{\cosh ml + \frac{h}{mK} \sinh ml} \right]$$

$$q_{fin} = kA_{cs}\, m\,(T_0 - T_a) \left[\frac{\sinh ml + \frac{h}{mk}\cosh ml}{\cosh ml + \frac{h}{mk}\sinh ml} \right]$$

$$q_{fin} = \sqrt{hPA_{cs}k}\,(T_0 - T_a) \left[\frac{\tanh ml + \frac{h}{mk}}{1 + \frac{h}{mk}\tanh ml} \right]$$