SUBJECT NAME : Heat Transfer

SUBJECT CODE : 3151909

Topic: General Heat Conduction Equation in spherical coordinate

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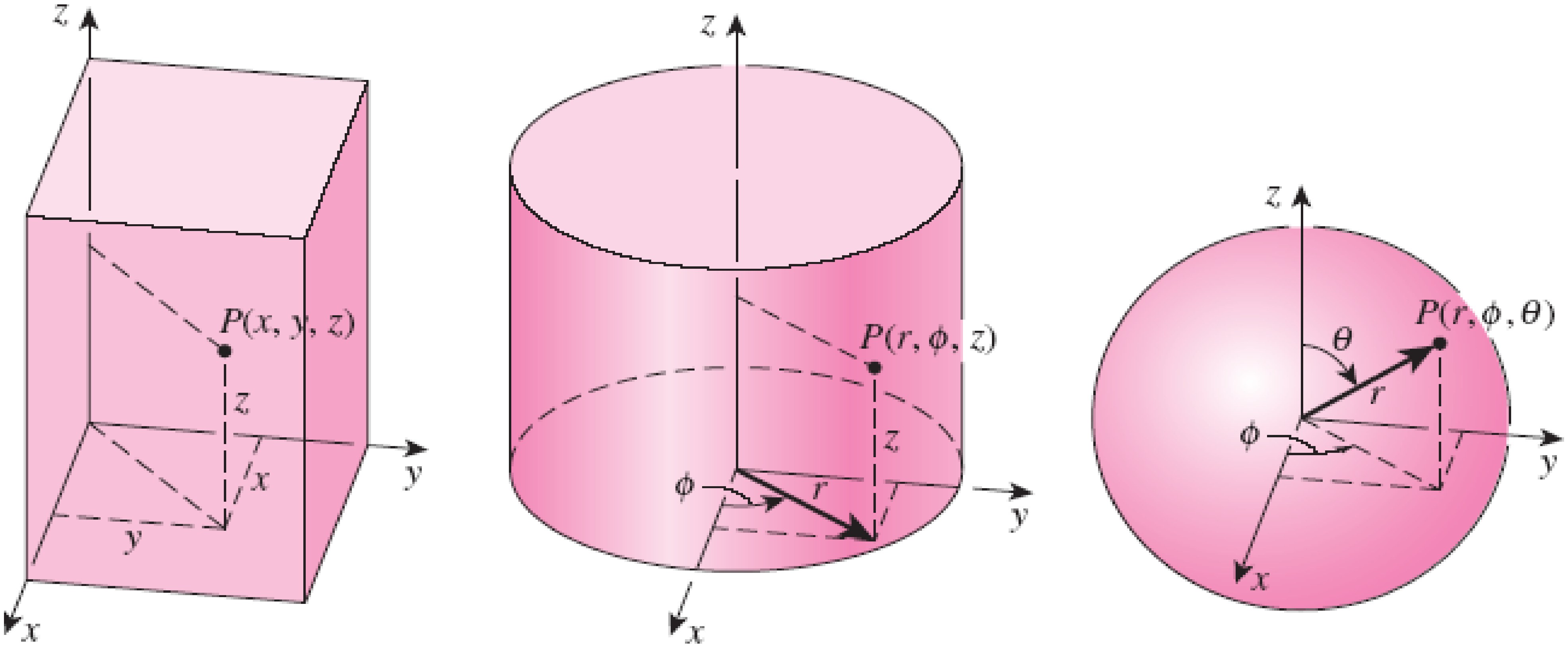
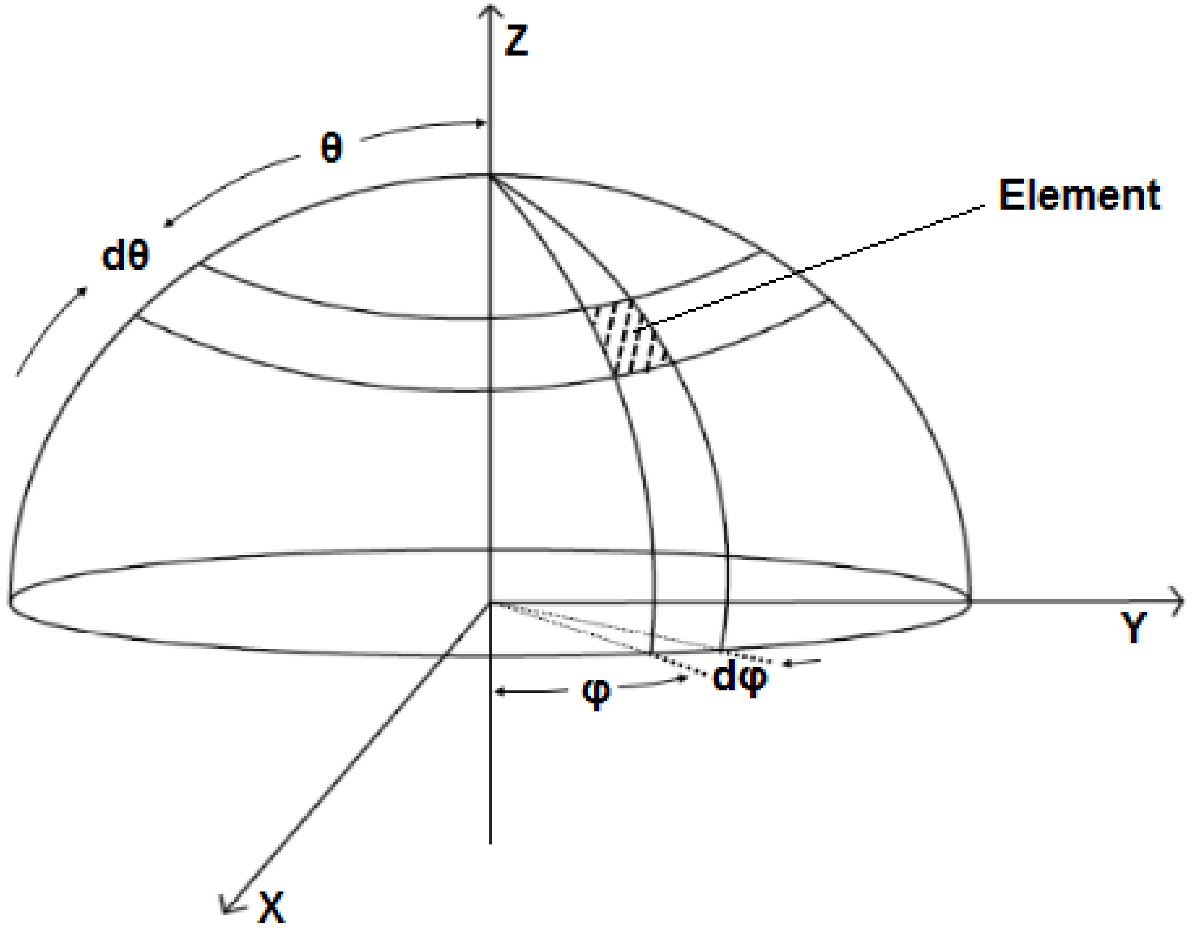
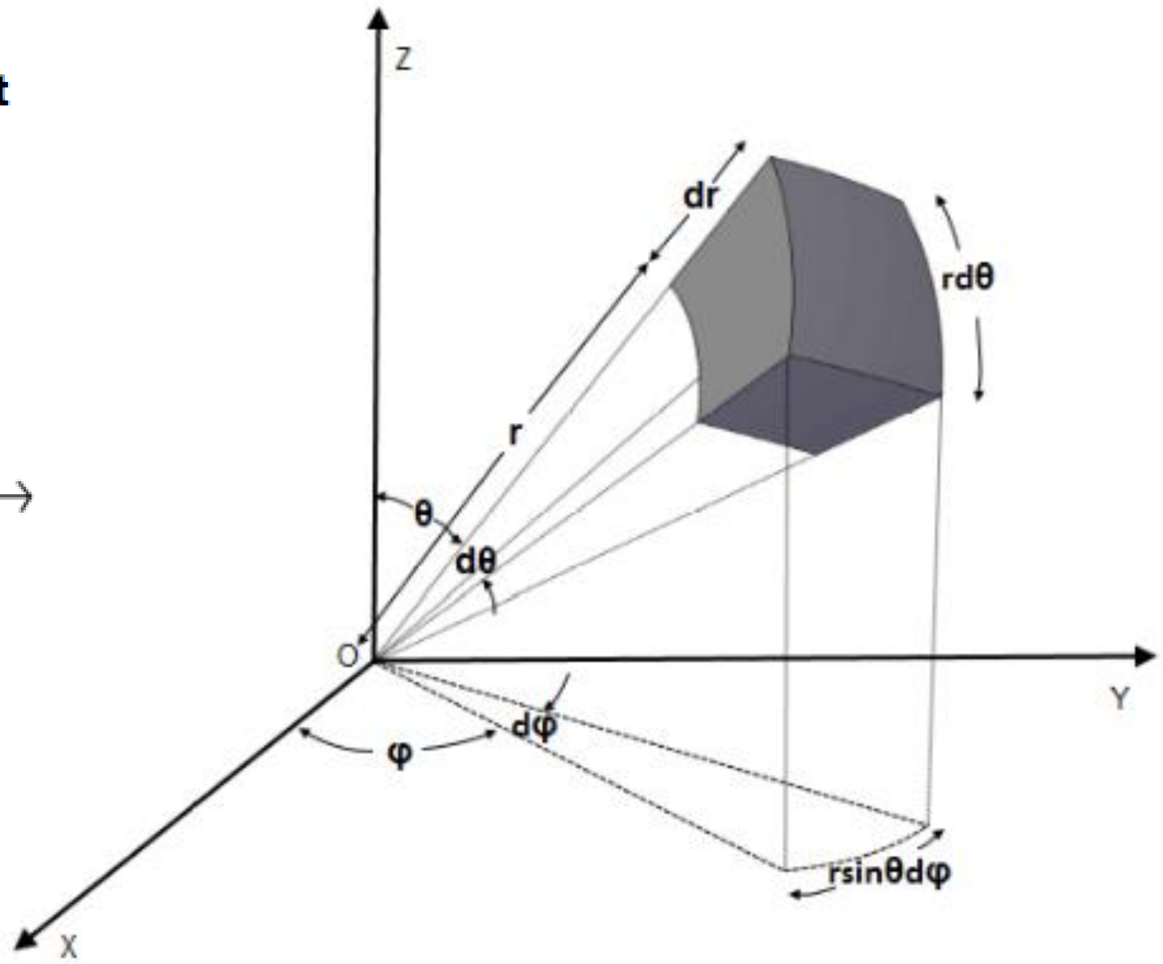
- Three prime coordinate systems:
  - rectangular $T(x, y, z, t)$
  - cylindrical $T(r, \phi, z, t)$
  - spherical $T(r, \phi, \theta, t)$.



(a) Rectangular coordinates (b) Cylindrical coordinates (c) Spherical coordinates


Z
θ
dθ
Element
Y
φ
dφ
X


Z
dr
rdθ
r
θ
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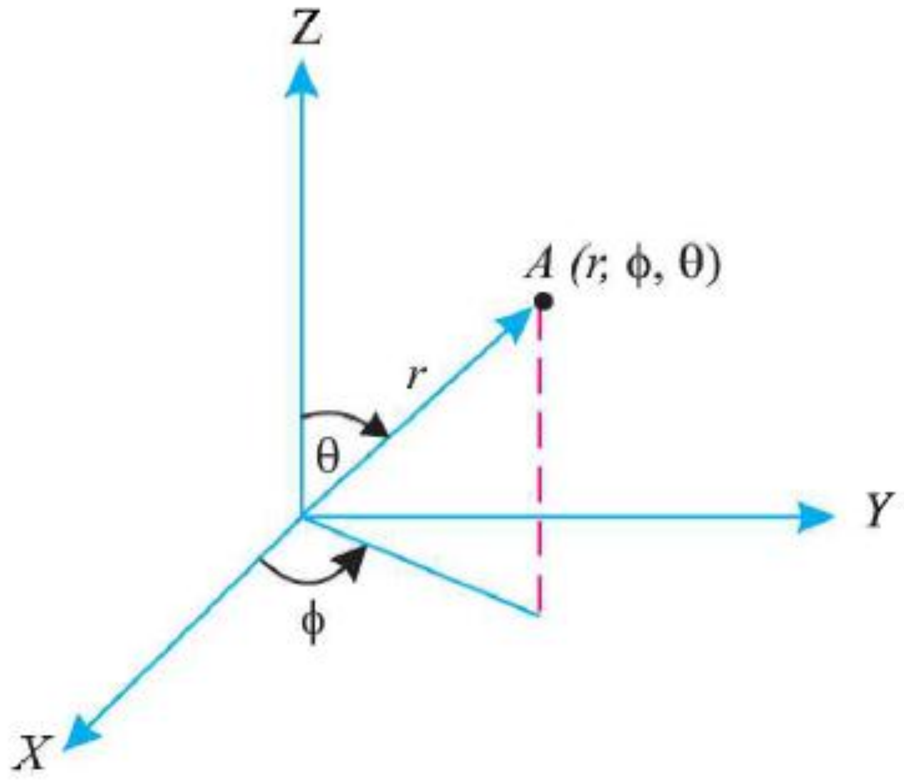
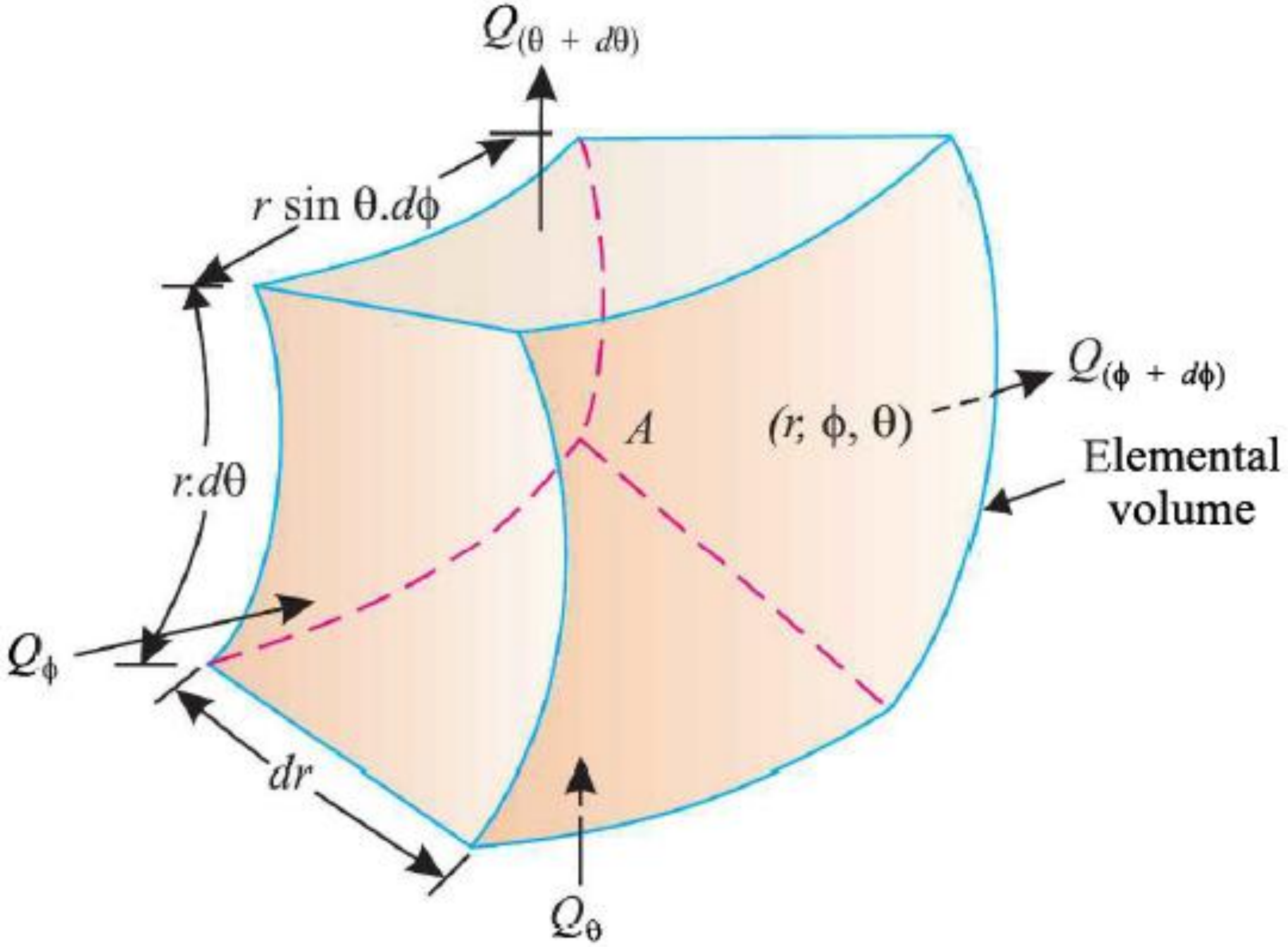
O
Y
dφ
φ
rsinθdφ
X

Let consider small element





- Consider an elemental volume having the coordinate (r,Ø,Θ) for three dimensional heat conduction analysis as shown in figure
-
- The volume of element = dr   rdΘ rsinΘ dØ
- Let m = mass of element
- ρ= density of element
- Qg = internal heat generation per unit time
- qg= internal heat generation per unit time per unit volume
- C= Specific heat
- K= thermal conductivity

Heat flow through r-Θ plane : Ø direction

$$Q'_\phi = -k\,(dr.rd\theta)\,\frac{\partial t}{r.\sin\theta.\partial\phi}\,d\tau$$

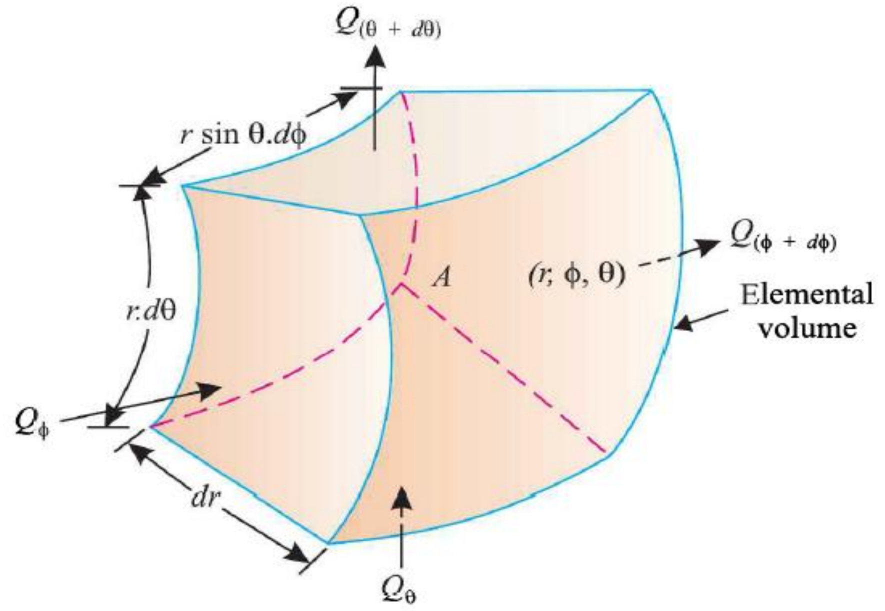
$$Q'_{(\phi+d\phi)} = Q'_\phi + \frac{\partial}{r.\sin\theta.\partial\phi}(Q'_\phi)\,r\sin\theta.d\phi$$

$$dQ'_\phi = Q'_\phi - Q'_{(\phi+d\phi)}$$

$$= -\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}(Q'_\phi)\,r\sin\theta.d\phi$$

$$= -\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\left[-k\,(dr.rd\theta)\,\frac{1}{r\sin\theta}\cdot\frac{\partial t}{\partial\phi}.\,d\tau\right]r\sin\theta.d\phi$$

$$= k\,(dr.rd\theta.r\sin\theta.d\phi)\,\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 t}{\partial\phi^2}\,d\tau$$



Heat flow through r- Ø plane : Θ direction

$$Q'_\theta = -k\,(dr.\,r\,\sin\,\theta.\,d\phi)\,\frac{\partial t}{r\partial\theta}\cdot d\tau$$

$$Q'_{(\theta+d\theta)} = Q'_\theta + \frac{\partial}{r\partial\theta}(Q'_\theta)\,rd\theta$$

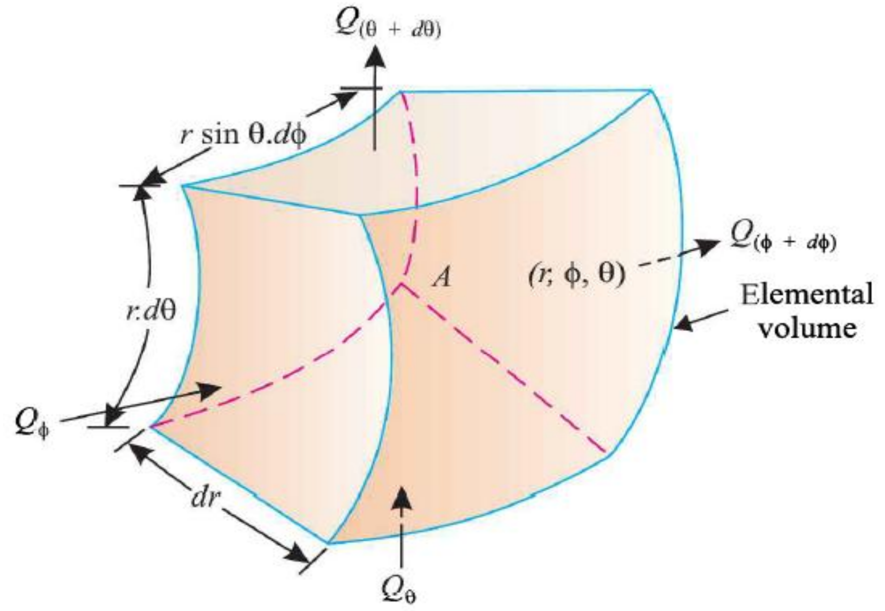
$$dQ'_\theta = Q'_\theta - Q'_{(\theta+d\theta)}$$

$$= -\frac{\partial}{r.\partial\theta}(Q'_\theta)\,r.d\theta$$

$$= -\frac{\partial}{r.\partial\theta}\left[-k\,(dr\,.r\,\sin\theta.d\phi)\,\frac{\partial t}{r.\partial\theta}.\,d\tau\right]r.d\theta$$

$$= \frac{k}{r}\,\frac{dr.rd\phi.rd\theta}{r}\,\frac{\partial}{\partial\theta}\left[\sin\,\theta\cdot\frac{\partial t}{\partial\theta}\right]d\tau$$

$$= k\,(dr.rd\theta.r\,\sin\,\theta.d\phi)\,\frac{1}{r^2\,\sin\,\theta}\cdot\frac{\partial}{\partial\theta}\left[\sin\,\theta\cdot\frac{\partial t}{\partial\theta}\right]d\tau$$



Heat flow through Θ - Ø plane : r direction

$$Q'_r = -k\,(rd\theta.r\sin\theta.d\phi)\,\frac{\partial t}{\partial r}\cdot\partial\tau$$

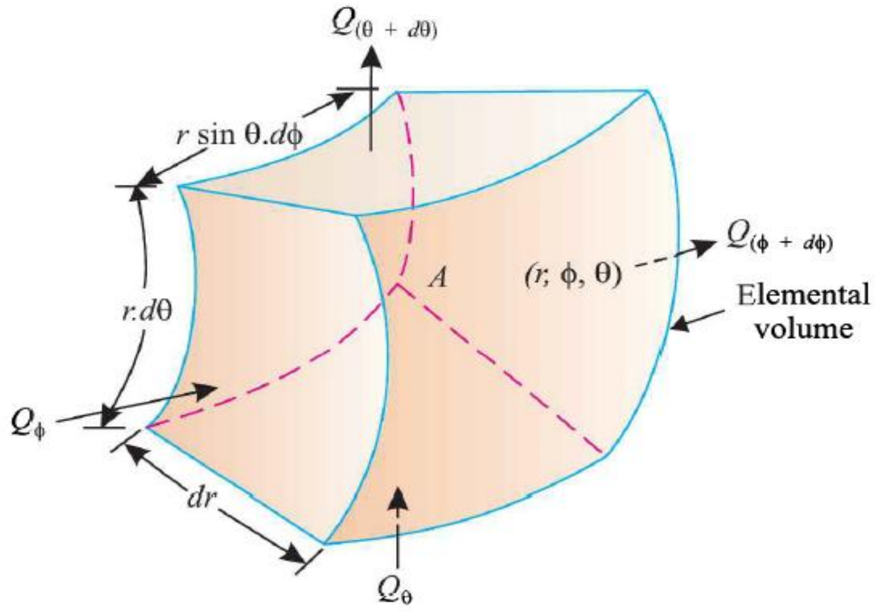
$$Q'_{(r+dr)} = Q'_r + \frac{\partial}{\partial r}(Q'_r)\,dr$$

$$dQ'_r = Q'_r - Q'_{(r+dr)}$$

$$= -\frac{\partial}{\partial r}(Q'_r)\,dr$$

$$= -\frac{\partial}{\partial r}\left[-k\,(rd\theta.r\sin\theta.d\phi)\,\frac{\partial t}{\partial r}\cdot d\tau\right]dr$$

$$= k\,d\theta.\sin\theta.d\phi\,dr\,\frac{\partial}{\partial r}\left[r^2\cdot\frac{\partial t}{\partial r}\right]d\tau$$



Net Heat Accumulated in the element

$$= k\ dr.rd\theta.r\ \sin\ \theta.d\phi \left[\frac{1}{r^2\ \sin^2\theta} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2\ \sin\ \theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta . \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 . \frac{\partial t}{\partial r} \right) \right] d\tau$$

B. *Heat generated within the element* (Q'_g) :

The total heat generated within the element is given by,

$$Q'_g = q_g\,(dr.rd\theta.r \sin\,\theta.d\phi)\ d\tau$$

C. *Energy stored in the element* :

The increase in thermal energy in the element is equal to

$$\rho\,(dr.rd\theta\ .\ r\ \sin\ \theta.d\phi)\ c.\ \frac{\partial t}{\partial \tau}\ .\ d\tau$$

Now, $(A) + (B) = (C)$ ...Energy balance/equation

$$\therefore \quad k\,dr.rd\theta.r\sin\theta.d\phi\left[\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 t}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta.\frac{\partial t}{\partial\theta}\right)+\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2.\frac{\partial t}{\partial r}\right)\right].d\tau$$

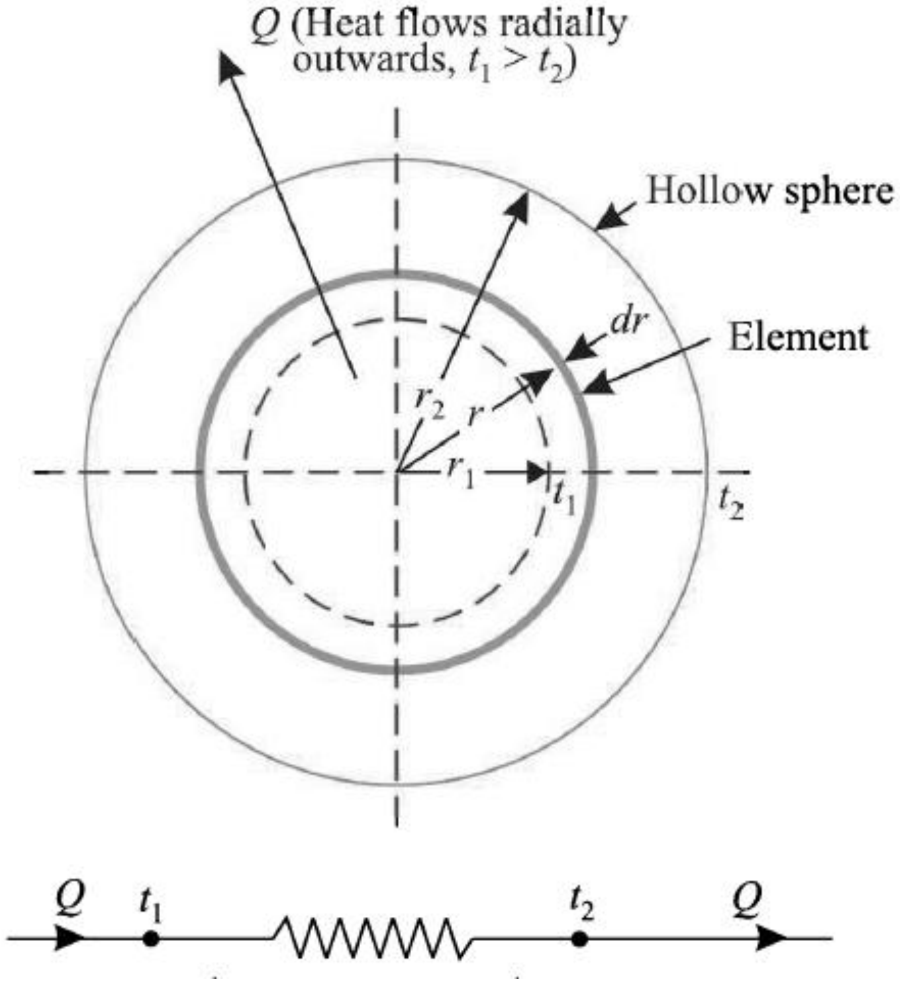
$$+\,q_g\,(dr.rd\theta.r\sin\theta.d\phi)\,d\tau=\rho(dr.rd\theta.r\sin\theta.d\phi)\,c.\frac{\partial t}{\partial\tau}.d\tau$$

Dividing both sides by $k.(dr.rd\theta.\ r\sin\theta.d\phi)d\tau$, we get

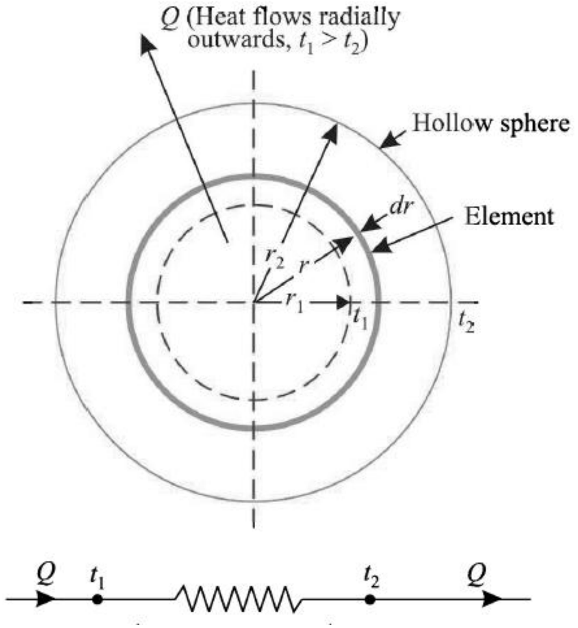
$$\left[\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 t}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\,.\,\frac{\partial t}{\partial\theta}\right)+\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\,.\,\frac{\partial t}{\partial r}\right)\right]+\frac{q_g}{k}$$

$$=\frac{\rho c}{k}\cdot\frac{\partial t}{\partial\tau}=\frac{1}{\alpha}\cdot\frac{\partial t}{\partial\tau}$$

Heat Conduction Through hollow and composite Spheres



Heat Conduction Through hollow and composite Spheres



Let, r_1, r_2 = Inner and outer radii,

t_1, t_2 = Temperatures of inner and outer surfaces, and

k = Constant thermal conductivity of the material with the given temperature range.

The general heat conduction equation in spherical coordinates is given as follows :

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)+\frac{1}{r^2\sin^2\phi}\cdot\frac{\partial^2 t}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right)+\frac{q_g}{k}=\frac{1}{\alpha}\cdot\frac{\partial t}{\partial\tau}$$

Heat Conduction Through hollow and composite Spheres

For steady state $\left(\frac{\partial t}{\partial \tau} = 0\right)$, unidirectional heat flow in the radial direction $\{t \neq f(\theta, \phi)\}$ and with no heat generation $(q_g = 0)$, the above equation reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \cdot \frac{dt}{dr}\right) = 0$$

or, $$\frac{d}{dr}\left(r^2 \cdot \frac{dt}{dr}\right) = 0 \quad \text{as} \quad \frac{1}{r^2} \neq 0$$

or, $$r^2 \cdot \frac{dt}{dr} = C \text{ (a constant)}$$

Integrating the above equation, we obtain

$$t = -\frac{C}{r} + C_1$$

(where C_1 = a constant of integration)

Using the following boundary conditions, we have

At $r = r_1, t = t_1$; At $r = r_2, t = t_2$

$\therefore$ $$t_1 = -\frac{C}{r_1} + C_1 \quad \text{...(i)}$$

$$t_2 = -\frac{C}{r_2} + C_1 \quad \text{...(ii)}$$

Heat Conduction Through hollow and composite Spheres

From (*i*) and (*ii*), we have

$$C = \frac{(t_1 - t_2)\, r_1\, r_2}{r_1 - r_2}$$

and,

$$C_1 = t_1 + \frac{(t_1 - t_2)\, r_1\, r_2}{r_1\,(r_1 - r_2)}$$

Substituting the values of these constants in eqn. (2.73), we get

$$t = -\frac{(t_1 - t_2)\, r_1\, r_2}{r\,(r_1 - r_2)} + t_1 + \frac{(t_1 - t_2)\, r_1\, r_2}{r_1\,(r_1 - r_2)}$$

$$t = -\frac{(t_1 - t_2)}{r\,(1/r_2 - 1/r_1)} + t_1 + \frac{(t_1 - t_2)}{r_1\,(1/r_2 - 1/r_1)}$$

$$t = t_1 + \frac{(t_1 - t_2)}{(1/r_2 - 1/r_1)}\left[\frac{1}{r_1} - \frac{1}{r}\right]$$

$$\frac{t - t_1}{t_2 - t_1} = \frac{1/r - 1/r_1}{1/r_2 - 1/r_1}$$

$$\frac{t - t_1}{t_2 - t_1} = \frac{r_2}{r}\left[\frac{r - r_1}{r_2 - r_1}\right] \qquad \text{[Dimensionless form]}$$

Heat Conduction Through hollow and composite Spheres

$$Q = -kA\frac{dt}{dr}$$

$$= -k.4\pi r^2 .\frac{d}{dr}\left[t_1 + \frac{(t_1 - t_2)}{(1/r_2 - 1/r_1)}\left(\frac{1}{r_1} - \frac{1}{r}\right)\right]$$

$$= -k.4\pi r^2 .\frac{t_1 - t_2}{(1/r_2 - 1/r_1)} \times -\left(-\frac{1}{r^2}\right)$$

$$= -k.4\pi r^2 .\frac{(t_1 - t_2)}{\left(\frac{r_1 - r_2}{r_1 . r_2}\right)} \times \frac{1}{r^2}$$

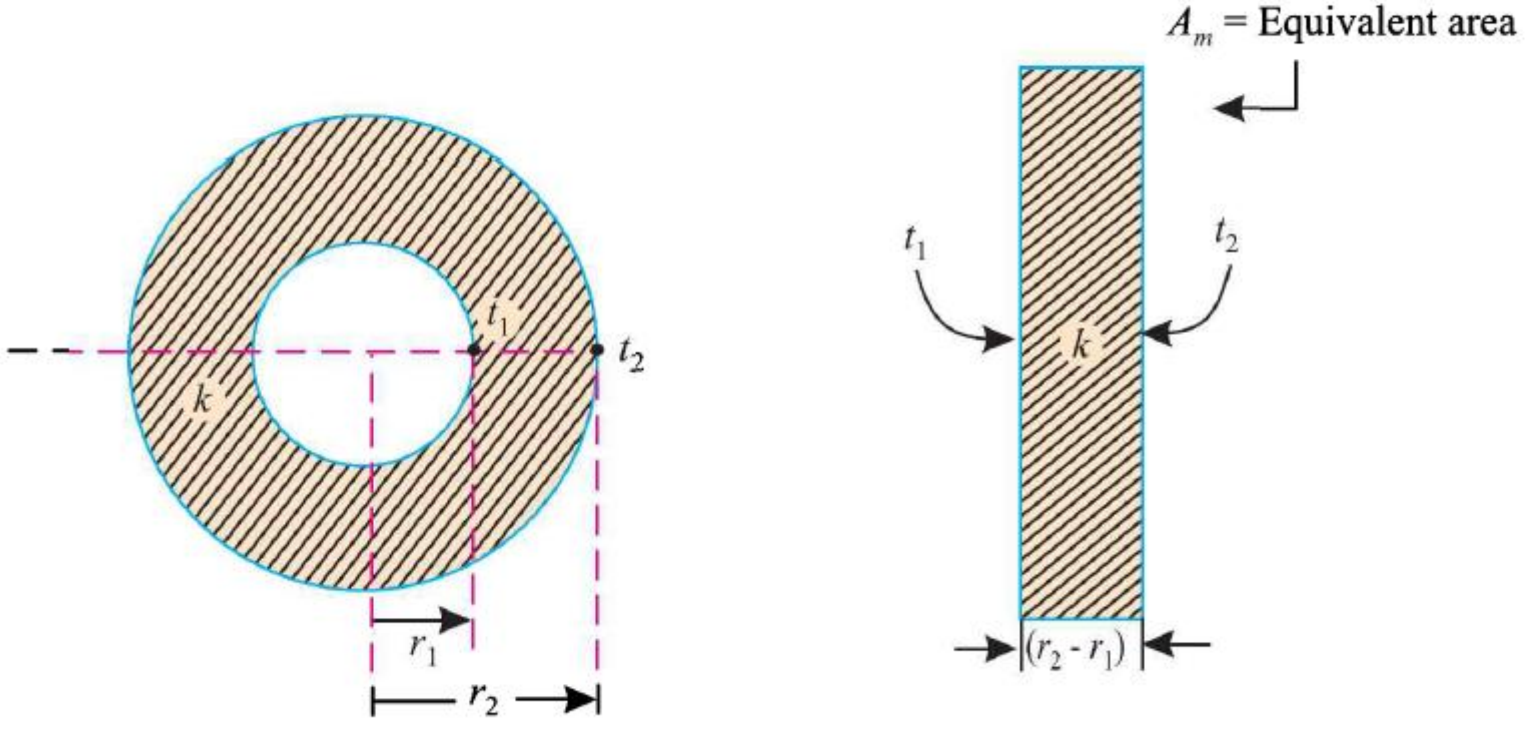
$$= -4\pi k\frac{(t_1 - t_2) r_1 r_2}{(r_1 - r_2)} = \frac{4\pi k (t_1 - t_2) r_1 r_2}{(r_2 - r_1)} = \frac{(t_1 - t_2)}{(r_2 - r_1)/4\pi k r_1 r_2}$$

i.e.

$$Q = \frac{(t_1 - t_2)}{\left[\frac{(r_2 - r_1)}{4\pi k r_1 r_2}\right]}\left[= \frac{\Delta t}{R_{th}}\right]$$

where the term $(r_2 - r_1) / 4\pi k r_1 r_2$ is the thermal resistance (R_{th}) for heat conduction through a hollow sphere.

Logarithmic Mean Area for the hollow cylinder



(*a*) Hollow cylinder (*b*) Plane wall

$$Q = \frac{(t_1 - t_2)}{\dfrac{\ln(r_2 / r_1)}{2\pi k L}}$$

$$Q = \frac{(t_1 - t_2)}{\dfrac{(r_2 - r_1)}{k A_m}}$$

A_m is so chosen that heat flow through cylinder and plane wall will be equal for the same thermal potential.

$$\therefore \qquad \frac{\dfrac{(t_1 - t_2)}{\text{In}\,(r_2 / r_1)}}{2\pi k L} = \frac{\dfrac{(t_1 - t_2)}{(r_2 - r_1)}}{k A_m}$$

$$\frac{\ln\,(r_2 / r_1)}{2\pi k L} = \frac{(r_2 - r_1)}{k A_m}$$

$$A_m = \frac{2\pi L (r_2 - r_1)}{\ln\,(r_2 / r_1)} = \frac{2\pi L r_2 - 2\pi L r_1}{\ln\,(2\pi L r_2 / 2\pi L r_1)}$$

$$A_m = \frac{A_0 - A_i}{\ln\,(A_0 - A_i)}$$