DIMENSIONAL ANALYSIS And Similarities

Unit -6

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Learning Objectives

1. Introduction to Dimensions & Units
2. Use of Dimensional Analysis
3. Dimensional Homogeneity
4. Methods of Dimensional Analysis
5. Rayleigh's Method

Learning Objectives

6. Buckingham’s Method

7. Model Analysis

8. Similitude

9. Model Laws or Similarity Laws

10. Model and Prototype Relations

Introduction

- Many practical real flow problems in fluid mechanics can be solved by using equations and analytical procedures. However, solutions of some real flow problems depend heavily on experimental data.

- Sometimes, the experimental work in the laboratory is not only time-consuming, but also expensive. So, the main goal is to extract maximum information from fewest experiments.

- In this regard, dimensional analysis is an important tool that helps in correlating analytical results with experimental data and to predict the prototype behavior from the measurements on the model.

Dimensions and Units

In dimensional analysis we are only concerned with the nature of the dimension i.e. its quality not its quantity.

- Dimensions are properties which can be measured.
  Ex.: Mass, Length, Time etc.,
- Units are the standard elements we use to quantify these dimensions.
  Ex.: Kg, Metre, Seconds etc.,

The following are the Fundamental Dimensions (MLT)

Mass	kg	M
Length	m	L
Time	s	T

Secondary or Derived Dimensions

Secondary dimensions are those quantities which posses more than one fundamental dimensions.

1.	Geometric				
	a)	Area	m^2		L^2
	b)	Volume	m^3		L^3
2.	Kinematic				
	a)	Velocity	m/s	L/T	$L.T^{-1}$
	b)	Acceleration	m/s^2	L/T^2	$L.T^{-2}$
3.	Dynamic				
	a)	Force	N	ML/T	$M.L.T^{-\cancel{1}2}$
	b)	Density	kg/m^3	M/L^3	$M.L^{-3}$

$F = ma$

$= \text{mass} \cdot m/s^2$

$= \frac{ML}{T^2} = M^1 L^1 T^{-2}$

(1) Geometric

- Area m^2 $M^0 L^2 T^0$
- Volume m^3 $M^0 L^3 T^0$
- Moment of inertia m^4 $M^0 L^4 T^0$
- Roughness m $M^0 L^1 T^0$

(2) Kinematics Quantities

- Velocity v m/s $M^0 L^1 T^{-1}$
- Angular Velocity ω rad/sec $M^0 L^0 T^{-1}$
- Rotational Speed N rpm $M^0 L^0 T^{-1}$
- Acceleration a m/s^2 $M^0 L^1 T^{-2}$
- Gravitational Acceleration g m/s^2 $M^0 L^1 T^{-2}$
- Kinematic Viscosity ν m^2/sec $M^0 L^2 T^{-1}$
- Discharge Q m^3/sec $M^0 L^3 T^{-1}$

(3) Dynamic Quantities

- Force, weight F, ω $F = mg$, $\omega = mg$ N $kg\,m/s^2$ $M^1 L^1 T^{-2}$
- Specific Weight ω N/m^3 $kg\,m/s^2 \cdot \frac{1}{m^3}$ $M^1 L^{-2} T^{-2}$
- Pressure $P = F/A$ N/m^2 $kg \cdot m/s^2 \cdot \frac{1}{m^2}$ $M^1 L^{-1} T^{-2}$
- Stress σ, τ F/A N/m^2 "
- Modulus of Elasticity E, K N/m^2 "
- Surface Tension σ N/m $kg\,m/s^2 \cdot \frac{1}{m}$ $M^1 L^0 T^{-2}$
- Density ρ kg/m^3 $M^1 L^{-3} T^0$
- Dynamic Viscosity μ $N \cdot s/m^2$ $M^1 L^{-1} T^{-1}$

(3) Dynamic Quantities

- Work, Energy J . (W, E) N.m kg m/s^2 . m $M L^2 T^{-2}$
- Power J/sec P N.m/sec kg . m/sec^2 . m/sec $M L^2 T^{-3}$
- Torque $T = F \cdot d$ N.m $M L^2 T^{-2}$
- Momentum M, P kg.m/s $M^1 L^1 T^{-1}$

$P = m \cdot v$

Use of Dimensional Analysis

CGS → MKS

1. Conversion from one dimensional unit to another
2. Checking units of equations (Dimensional Homogeneity)
3. Defining dimensionless relationship using
   a) Rayleigh's Method
   b) Buckingham's π-Theorem
4. Model Analysis

Dimensional Homogeneity

Dimensional Homogeneity means the dimensions in each equation on both sides equal.

$V = \sqrt{2gH}$

Let us consider the equation, $V = \sqrt{2gH}$

Dimension of L.H.S. $= V = \frac{L}{T} = LT^{-1}$

Dimension of R.H.S. $= \sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T} = LT^{-1}$

Dimension of L.H.S. $=$ Dimension of R.H.S. $= LT^{-1}$

$\therefore$ Equation $V = \sqrt{2gH}$ is dimensionally homogeneous. So it can be used in any system of units.

Rayeligh’s Method

To define relationship among variables

This method is used for determining the expression for a variable which depends upon maximum <u>three or four</u> variables only.

MLT

Rayeligh's Method

Methodology:

$X = f(x_1, x_2 x_3) = \phi(x_1 x_2 x_3)$

Let X is a function of X_1, X_2, X_3 and mathematically it can be written as $X = f(X_1, X_2, X_3)$

$X = K\, x_1^a\, x_2^b\, x_3^c$ MLT

This can be also written as

$X = K\,(X_1^a, X_2^b, X_3^c)$ where K is constant and a, b and c are arbitrarily powers

The values of a, b and c are obtained by comparing the powers of the fundamental dimension on both sides.

Rayeligh’s Method

Problem: Find the expression for Discharge Q in a open channel flow when Q is depends on Area A and Velocity V.

Solution:

$Q = K.A^a.V^b \rightarrow 1$

where K is a Non-dimensional constant

Substitute the dimensions on both sides of equation 1

$M^0 L^3 T^{-1} = K. (L^2)^a.(LT^{-1})^b$

Equating powers of M, L, T on both sides,

Power of T, $-1 = -b \Rightarrow b=1$

Power of L, $3= 2a+b \Rightarrow 2a = 2-b = 2-1 = 1$

Substituting values of a, b, and c in Equation 1m

$Q = K. A^1. V^1 = V.A$

$Q = f(A, V)$

$Q = \underline{k} A^1 V^1$

$Q = K A^a V^b$

$\boxed{Q = AV}$

$Q \quad m^3/s \quad M^0 L^3 T^{-1}$

$A \quad m^2 \quad M^0 L^2 T^0$

$V \quad m/s \quad M^0 L^1 T^{-1}$

$M^0 L^3 T^{-1} = K . L^{2a} . L^b T^{-b}$

$M^0 L^3 T^{-1} = K \ L^{2a+b} \ T^{-b}$

$2a+b = 3 \qquad -b = -1$

$a = 1 \qquad b = 1$

Problem : Find the equation for the power developed by a pump if it depends on head H discharge Q and specific weight γ of the fluid.

Solution:

Power J/s W		$= L^2MT^{-3}$
Head m		$= L^1M^oT^o$
Discharge m^3/s		$= L^3M^oT^{-1}$
Specific Weight		$= L^{-2}MT^{-2}$

$P = f(H, Q, \gamma)$

$P = K \cdot H^a \cdot Q^b \cdot \gamma^c$

$[P] = [H]^a \cdot [Q]^b \cdot [\gamma]^c$

$[L^2MT^{-3}] = [LM^oT^o]^a \cdot [L^3M^oT^{-1}]^b \cdot [L^{-2}M^1T^{-2}]^c$

Equating the powers of M, L and T on both sides,

Power of M, $\quad 1 = c$

Power of T, $\quad -3 = -b - 2c$ or $b = -2 + 3$ or $\mathbf{b = 1}$

Power of L, $\quad 2 = a + 3b - 2c$ or $2 = a + 3 - 2$ or $\mathbf{a = 1}$

Substituting the values of a, b and c

$P = K \cdot H^1 \cdot Q^1 \cdot \gamma^1$

$P = K \cdot H \cdot Q \cdot \gamma$ When, $K = 1$ $\mathbf{P = H \cdot Q \cdot \gamma}$

$P = f(H, Q, \gamma)$

$P = K\, H^a\, Q^b\, \gamma^c$

$P = K\, H^1\, Q^1\, \gamma^1$

$K = 1$

$\boxed{P = H \cdot Q \cdot \gamma}$

Problem _ Find an expression for drag force R on a smooth sphere of diameter D moving with uniform velocity V in a fluid of density ρ and dynamic viscosity μ..

Solution:

Force	$= LMT^{-2}$
Diameter	$= LM^oT^o$
Velocity	$= LM^oT^{-1}$
Mass density	$= L^3MT^o$

$R = f(D, V, \rho, \mu)$

$R = K \cdot D^a \cdot V^b \cdot \rho^c, \mu^d$

$[R] = [D]^a \cdot [V]^b \cdot [\rho]^c \cdot [\mu]^d$

$[LMT^{-2}] = [LM^oT^o]^a \cdot [LM^oT^{-1}]^b \cdot [L^{-3}MT^o]^c \cdot [L^{-1}MT^{-1}]^d$

Equating the powers of M, L and T on both sides,

Power of M, $1 = c + d$ or $\mathbf{c = 1 - d}$

Power of T, $-2 = -b - d$ or $\mathbf{b = 2 - d}$

Power of L, $1 = a + b - 3c - d$ or $1 = a + 2 - d - 3(1 - d) - d$

$1 = a + 2 - d - 3 + 3d - d$ or $\mathbf{a = 2 - d}$

Substituting teh values of a, b, and c

$$R = K \cdot D^{2-d} \cdot V^{2-d} \cdot \rho^{1-d}, \mu^d = K \frac{D^2}{D^d} \cdot \frac{V^2}{V^d} \cdot \frac{\rho}{\rho^d} \cdot \mu^d$$

$$. = K \cdot \rho V^2 D^2 \left[\frac{\mu}{\rho VD}\right]^d = \rho V^2 D^2 \phi \left[\frac{\mu}{\rho VD}\right] = \rho V^2 D^2 \phi \left[\frac{\rho VD}{\mu}\right]$$

Buckingham's π-Theorem

This method of analysis is used when number of variables are more.

Theorem:

7 ③

If there are n variables in a physical phenomenon and those n variables contain m dimensions, then variables can be arranged into (n-m) dimensionless groups called π terms.

7 - 3 = 4 $\pi_1, \pi_2, \pi_3, \pi_4,$

Explanation:

If $f(X_1, X_2, X_3, \ldots\ldots\ldots X_n) = 0$ and variables can be expressed using m dimensions then $f(\pi_1, \pi_2, \pi_3, \ldots\ldots\ldots \pi_{n-m}) = 0$ where, $\pi_1, \pi_2, \pi_3, \ldots$ are dimensionless groups.

Each π term contains (m + 1) variables out of which m are of repeating type and one is of non-repeating type.

Each π term being dimensionless, the dimensional homogeneity can be used to get each π term.

π denotes a non-dimensional parameter

Buckingham's π-Theorem

$n = 7$ π Term $= n - m = 7 - 3 = 4$ m 1

$m = 3$

π1 $x_1\ x_2\ x_3\ x_4$

π2 $x_1\ x_2\ x_3\ x_5$

π3 $x_1\ x_2\ x_3\ x_6$

π4 $x_1\ x_2\ x_3\ x_7$

$m+1$

Selecting Repeating Variables:

1. Avoid taking the quantity required as the repeating variable.
2. Repeating variables put together should not form dimensionless group.
3. No two repeating variables should have same dimensions.
4. Repeating variables can be selected from each of the following properties.
   - Geometric property → Length, height, width, area
   - Flow property → Velocity, Acceleration, Discharge
   - Fluid property → Mass density, Viscosity, Surface tension

Problem 1 *The pressure difference Δp in a pipe of diameter D and length l due to viscous flow depends on the velocity V, viscosity μ and density ρ. Using Buckingham's π-theorem, obtain an expression for Δp.*

$\Delta P = F(D, l, V, \mu, \rho)$

Solution.

Δp is a function of D, l, V, μ, ρ or $\Delta p = f(D, l, V, \mu, \rho)$

$F(\Delta P, D, l, V, \mu, \rho) = 0$

1 2 3 4 5 6 $\quad n = 6$

or $\quad f_1(\Delta p, D, l, V, \mu, \rho) = 0$ ...(*i*)

Total number of variables, $n = 6$

Number of fundamental dimension, $m = 3$

$\boxed{m = 3}$ $\quad n - m = 6 - 3 = 3$

Number of π-terms $= n - 3 = 6 - 3 = 3$

$f_1(\pi_1, \pi_2, \pi_3) = 0$

Hence equation (*i*) is written as $f_1(\pi_1, \pi_2, \pi_3) = 0$ ...(*ii*)

$m+1$

Each π-term contains $m + 1$ variables, *i.e.*, $3 + 1 = 4$ variable. Out of four variables, three are repeating variables.

Choosing D, V, μ as repeating variables, we have π-terms as

$$\pi_1 = D^{a_1} . V^{b_1} . \mu^{c_1} . \Delta p$$

$$\pi_2 = D^{a_2} . V^{b_2} . \mu^{c_2} . l$$

$$\pi_3 = D^{a_3} . V^{b_3} . \mu^{c_3} . \rho$$

First π-term $\pi_1 = D^{a_1} . V^{b_1} . \mu^{c_1} . \Delta p$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_1} . (LT^{-1})^{b_1} . (ML^{-1}T^{-1})^{c_1} . ML^{-1}T^{-2}.$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_1 + 1$, $\therefore\ c_1 = -1$

Power of L, $0 = a_1 + b_1 - c_1 - 1$, $\therefore\ a_1 = -b_1 + c_1 + 1 = 1 - 1 + 1 = 1$

Power of T, $0 = -b_1 - c_1 - 2$, $\therefore\ b_1 = -c_1 - 2 = 1 - 2 = -1$

Substituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = D^1 . V^{-1} . \mu^{-1} . \Delta p = \frac{D\Delta p}{\mu V}.$$

Second π-term $\pi_2 = D^{a_2} . V^{b_2} . \mu^{c_2} . l$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} . (LT^{-1})^{b_2} . (ML^{-1}T^{-1})^{c_2} . L.$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_2$, $\therefore\ c_2 = 0$

Power of L, $0 = a_2 + b_2 - c_2 + 1$, $\therefore\ a_2 = -b_2 + c_2 - 1 = -1$

Power of T, $0 = -b_2 - c_2$, $\therefore\ b_2 = -c_2 = 0$

Substituting the values of a_2, b_2 and c_2 in π_2,

$$\pi_2 = D^{-1} . V^0 . \mu^0 . l = \frac{l}{D}.$$

$\Pi_1 = D^{a_1}\ V^{b_1}\ \mu^{c_1}\ \Delta P$

$M^0 L^0 T^0 = (L^1)^{a_1} (L^1 T^{-1})^{b_1} (M^1 L^{-1} T^{-1})^{c_1} M^1 L^{-1} T^{-2}$

$a_1 = 1$

$b_1 = -1$

$c_1 = -1$

$\Pi_1 = D^1 V^{-1} \mu^{-1} \Delta P$

$$\boxed{\Pi_1 = \frac{D \Delta P}{V \mu}}$$

$f_1(\pi_1 \; \pi_2 \pi_3) = 0$

Third π-term $\pi_3 = D^{a_3} . V^{b_3} . \mu^{c_3} . \rho$

Substituting the dimension on both sides,

$$M^0L^0T^0 = L^{a_3} . (LT^{-1})^{b_3} . (ML^{-1}T^{-1})^{c_3} . ML^{-3}.$$

Equating the powers of M, L, T on both sides

Power of M, $\quad 0 = c_3 + 1, \quad \therefore \; c_3 = -1$

Power of L, $\quad 0 = a_3 + b_3 - c_3 - 3, \quad \therefore \; a_3 = -b_3 + c_3 + 3 = -1 - 1 + 3 = 1$

Power of T, $\quad 0 = -b_3 - c_3, \quad \therefore \; b_3 = -c_3 = -(-1) = 1$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = D^1 . V^1 . \mu^{-1} . \rho = \frac{\rho DV}{\mu}.$$

Substituting the values of π_1, π_2 and π_3 in equation (*ii*),

$$\boxed{f_1\left(\frac{D\Delta p}{\mu V}, \frac{l}{D}, \frac{\rho DV}{\mu}\right) = 0} \quad \text{or} \quad \frac{D\Delta p}{\mu V} = \phi\left[\frac{l}{D}, \frac{\rho DV}{\mu}\right] \quad \text{or} \quad \boxed{\Delta p = \frac{\mu V}{D}\phi\left[\frac{l}{D}, \frac{\rho DV}{\mu}\right]}$$

Experiments show that the pressure difference Δp is a linear function $\frac{l}{D}$. Hence $\frac{l}{D}$ can be taken out of the functional as

$$\Delta p = \frac{\mu \mathbf{V}}{\mathbf{D}} \times \frac{\mathbf{L}}{\mathbf{D}} \phi\left[\frac{\rho \mathbf{DV}}{\mu}\right]. \textbf{ Ans.}$$

Expression for difference of pressure head for viscous flow

$$h_f = \frac{\Delta p}{\rho g} = \frac{\mu V}{D} \times \frac{l}{D} \times \frac{1}{\rho g} \phi [R_e] \qquad \left\{\because \; \frac{\rho DV}{\mu} = R_e\right\}$$

$$= \frac{\mu \mathbf{VL}}{\mathbf{wD}^2} \phi\, [\mathbf{R_e}]. \textbf{ Ans.}$$